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Finite Groups of Plane Birational Transformations with Eight Fundamental Points.

By F. R. Sharpe.

The enumeration of the finite groups of plane birational transformations is due to S. Kantor and A. Wiman. The former, in his Naples Prize Memoir,* showed that all periodic plane birational transformations could be reduced by combinations of quadratic transformations into one of a certain number of types having at most 8 fundamental points. The groups of periodic transformations having 3, 4, 5 or 6 fundamental points have been completely determined.† In the cases of 6, 7 or 8 fundamental points the Grassmann depiction of the plane on a cubic surface may be used to advantage. The groups of periodic transformations with 6 fundamental points are therefore isomorphic with the groups of linear transformations of the 27 lines of a cubic surface. In the cases of 7 or 8 fundamental points the cubic surface may be depicted upon a double plane by means of a certain (1, 2) correspondence. In the case of 8 fundamental points the curve of branch points on the double plane is a sextic with 2 coincident triple points.‡ A. Wiman § found the equation of this curve and determined its groups of transformations. The purpose of this paper (suggested by Prof. Virgil Snyder) is to determine the transformations in the simple plane in which the 8 fundamental points lie that correspond to a given group of transformations in the double plane. There are 120 conics that are tangent to the sextic at the triple points and also at 3 other points.|| It is shown that each conic leads to the partial determination of the cubic surface. The complete determination of the cubic surface then reduces to finding the bitangents to a certain quartic curve. To the triple infinity of conics that are tangent to the sextic at the triple point, correspond a triple infinity of quadrics that are tangent to the cubic surface at two points O, O'. This system of quadrics meets the cubic surface in space-sextic curves. To the 120 special conics in the double plane cor-

* "Prémiers fondaments pour un théorie des transformations périodiques univoques," 1891.

† A. Wiman, *Math. Annalen,* Band 48.

‡ S. Kantor, "Theorie der endlichen Gruppen," 1895.

§ *Loc. cit.*

|| Schottky, *Crelle,* Band 103.

respond 120 composite space sextics. Of these, 27 are the 27 lines on the cubic surface and 27 residual quintics; 2 are the points O, O' and 2 sextics; 54 are the residual conics found by passing planes through the 27 lines and O or O' and 54 residual quartics. In depicting the cubic surface on the simple plane, any line may be depicted as a point a_1. Any of the 15 lines skew to a_1 may be depicted as a conic b_1, not passing through a_1. Five other lines skew to a_1, but meeting b_1, are depicted as points a_2, a_3, a_4, a_5, a_6 on the conic b_1. O and O' are depicted as points, $a_1, a_2, a_3, a_4, a_5, a_6, O, O'$ being the 8 fundamental points. Given the transformation in the double plane, the transformation on the cubic surface can be found. There are two cases according as we do or do not interchange the two sheets of the double plane. To this interchange corresponds the interchange of the 2 parts of each composite sextic of the cubic surface and the Bertini transformation of order 17 in the simple plane. We can now determine the images of 6 a lines and of O and O' on the cubic surface, and hence of the 8 fundamental points in the simple plane. For a special case, the determination of the 120 conics, the cubic surface and the transformation in the simple plane is actually carried out.

By using homogeneous coordinates (x, y, z, u), letting O be $(0, 0, 1, 0)$ and O' $(0, 0, 0, 1)$, the equation of the cubic surface may be reduced to the form

$$z(zu + f_2) + u(zu + \phi_2) + zuf_1 + f_3 = 0, \tag{1}$$

where f_2, ϕ_2, f_1, f_3 are homogeneous functions of x and y of degrees $2, 2, 1, 3$ respectively. The quadric surfaces tangent to (1) at O, O' are

$$zu = Ax^2 + 2Hxy + By^2 = F_2. \tag{2}$$

We may write (1) in the form

$$\begin{aligned}\{z(zu+f_2)-u(zu+\phi_2)\}^2 &= (zuf_1+f_3)^2-4zu(zu+f_2)(zu+\phi_2)\\ &= 4\left(-zu-\frac{4f_2+4\phi_2-f_1^2}{12}\right)^3-\left(-zu-\frac{4f_2+4\phi_2-f_1^2}{12}\right)\left\{2f_1f_3-4f_2\phi_2+\frac{(4f_2+4\phi_2-f_1^2)^2}{12}\right\}\\ &\quad-\left\{-f_3^2+\frac{(2f_1f_3-4f_2\phi_2)(4f_2+4\phi_2-f_1^2)}{12}+\frac{(4f_2+4\phi_2-f_1^2)^3}{216}\right\}.\end{aligned} \tag{3}$$

For a given value of $\frac{y}{x}$, (1) and (2) give 2 sets of values for $\frac{z}{x}, \frac{u}{x}$. If we set up a correspondence between a point (x, y, z, u) on the cubic surface and a point (x', y', z') on a plane, where

$$x' = x, \quad y' = y, \quad z'y' = -zu - \frac{4f_2 + 4\phi_2 - f_1^2}{12}, \tag{4}$$

any plane through O, O' will meet the curve of intersection of (1) and (2) in 2 points that will be represented on the plane by the same point. The plane is therefore a double plane.

The 2 points coincide if

$$z\,(z\,u + f_2) - u\,(z\,u + \phi_2) = 0. \tag{5}$$

The curve of degree 9 in which (5) meets (1) is represented on the double plane by the sextic curve of branch points

$$4\,z'^3\,y'^3 - z'\,y'\,f_4 - f_6 = 0, \tag{6}$$

where, from (3),

$$f_4 = 2\,f_1\,f_3 - 4\,f_2\,\phi_2 + \frac{(4\,f_2 + 4\,\phi_2 - f_1^2)^2}{12},$$

$$f_6 = -\,f_3^2 + \frac{(2\,f_1\,f_3 - 4\,f_2\,\phi_2)\,(4\,f_2 + 4\,\phi_2 - f_1^2)}{12} + \frac{(4\,f_2 + 4\,\phi_2 - f_1^2)^3}{216}. \tag{7}$$

From (2) and (4) the sextic curves of intersection of (1) and (2) are represented on the double plane by conics

$$z'\,y' = F_2' = p\,x'^2 + q\,x'\,y' + r\,y'^2, \tag{8}$$

that are tangent to (6) at (0, 0, 1).

There are 120 sets of values of p, q, r such that, when (8) is substituted in (6), the left-hand member of (6) is of the form

$$(\alpha\,x'^3 + \beta\,x'^2\,y' + \gamma\,x'\,y'^2 + \delta\,y'^3)^2.$$

The conic (8) is then tangent to (6) at 3 points as well as at (0, 0, 1).

Substituting any 1 of these sets of values of p, q, r in (8), let

$$\frac{4\,f_2 + 4\,\phi_2 - f_1^2}{12} = -\,F_2'. \tag{9}$$

From (7) it follows that

$$\begin{aligned} f_3^2 &= -\,f_6 - (f_4 - 12\,F_2'^2)\,F_2' - 8\,F_2'^3 \\ &= 4\,F_2'^3 - f_4\,F_2' - f_6 \\ &= (\alpha\,x'^3 + \beta\,x'^2\,y' + \gamma\,x'\,y'^2 + \delta\,y'^3)^2. \end{aligned}$$

Hence,

$$f_3 = \alpha\,x^3 + \beta\,x^2\,y + \gamma\,x\,y^2 + \delta\,y^3. \tag{10}$$

From (7) and (9) we find

$$(f_2 - \phi_2)^2 = \left(\frac{f_1^2}{4} - 3\,F_2'\right)^2 - 2\,f_1\,f_3 - 12\,F_2'^2 + f_4. \tag{11}$$

If $z' = \dfrac{f_1}{2}$ is a bitangent to the quartic

$$(z'^2 - 3\,F_2')^2 - 4\,z'\,f_3 - 12\,F_2'^2 + f_4 = 0, \tag{12}$$

then

$$(f_2 - \phi_2)^2 = (s\,x'^2 + t\,x'\,y' + v\,y^2)^2, \tag{13}$$

and f_2, ϕ_2 can be found from (9) and (13) in the form

$$\frac{f_1^2}{8} - \frac{3}{2}F_2' \pm \frac{1}{2}\,(s\,x'^2 + t\,x'\,y' + v\,y'^2). \tag{14}$$

It is to be remarked that f_4 and $-f_6$ are the invariants I and J of (12), regarded as a quartic in z', and that (6), regarded as a cubic in $z'y'$, is the reducing cubic of (12).

The determination of the 120 tritangent conics to (6) appears to be a matter of considerable difficulty. In the particular case $f_4 \equiv 0$, $f_6 \equiv -x'^6 - y'^6$ we have to determine p, q, r so that the substitution in (6) of

$$\sqrt[3]{4}\, z'y' = p x'^2 + q x'y' + r y'^2$$

gives (dropping the primes)

$$(p x^2 + q x y + r y^2)^3 + x^6 + y^6 = (\alpha x^3 + \beta x^2 y + \gamma x y^2 + r y^3)^2. \qquad (15)$$

The solutions are of 5 types:

$$p x^2 + q x y + r y^2 = -x^2 \text{ or } -y^2, \qquad \text{(I)}$$

and 4 other solutions found by substituting ϵx for x, ϵy for y, where $\epsilon^6 = 1$.

$$p x^2 + q x y + r y^2 = \sqrt[3]{2}\, x y, \qquad \text{(II)}$$

and 5 other solutions found by the same substitution.

$$p x^2 + q x y + r y^2 = -x^2 - y^2 \sqrt[3]{4} \text{ or } -y^2 - x^2 \sqrt[3]{4}, \qquad \text{(III)}$$

and 16 others found as in the previous types.

$$p x^2 + q x y + r y^2 = \sqrt[3]{2}\, \{x y + k (x+y)^2\}, \qquad \text{(IV)}$$

where $k = -2 - \sqrt[3]{2} - \sqrt[3]{4}$, $-2 - \epsilon^2 \sqrt[3]{2} - \epsilon^4 \sqrt[3]{4}$, $-2 - \epsilon^4 \sqrt[3]{2} - \epsilon^2 \sqrt[3]{4}$, and 53 others of the same type. This result is obtainable by putting $\alpha = \delta$, $\beta = \gamma$ so that $(x+y)^2$ is a factor of the right-hand member of (15). We may now take

$$p x^2 + q x y + r y^2 = \sqrt[3]{2}\, \{x y + k (x+y)^2\},$$

since $p = r$ and $(x+y)^2$ is a factor, where k is chosen so that

$$\{(x+y)^2 - 3 x y\}^2 + 2\, \{k^3 (x+y)^4 + 3 k^2 (x+y)^2 x y + 3 k x^2 y^2\}$$

is a perfect square. Hence,

$$9 (k^2 - 1)^2 = (1 + 2 k^3)(9 + 6 k);$$

that is,

$$k^3 + 6 k^2 + 6 k + 2 = 0,$$

which gives the stated values for k.

The remaining 36 solutions are of a more general type. From (15) we have

$$\begin{aligned} &\alpha^2 = p^3 + 1, \quad && 2\alpha\beta = 3 p^2 q, \quad && 2\alpha\gamma + \beta^2 = 3 p^2 r + 3 p q^2, \\ &\beta^2 = r^3 + 1, && 2\gamma\delta = 3 r^2 q, && 2\beta\delta + \gamma^2 = 3 r^2 p + 3 r q^2, \end{aligned}$$
$$2\alpha\delta + 2\beta\gamma = q^3 + 6 p q r.$$

Eliminating α and δ,

$$\left.\begin{array}{ll} 4\,(p^3+1)\,\beta^2 = 9\,p^4 q^2, & 4\,(p^3+1)\,\gamma^2 = (3\,p^2 r + 3\,p\,q^2 - \beta^2)^2, \\ 4\,(r^3+1)\,\gamma^2 = 9\,r^4 q^2, & 4\,(r^3+1)\,\beta^2 = (3\,p^2 r + 3\,r\,q^2 - \gamma^2)^2, \\ \multicolumn{2}{c}{4\,(p^3+1)\,(r^3+1) = (q^3 + 6\,p\,q\,r - 2\,\beta\,\gamma)^2.} \end{array}\right\} \quad (16)$$

Eliminating β and γ,

$$\left.\begin{array}{l} 16\,(p^3+1)^3 r^4 q^2 = (r^3+1)\,\{4\,p\,(p\,r+q^2)\,(p^3+1) - 3\,p^4 q^2\}, \\ 16\,(r^3+1)^3 p^4 q^2 = (p^3+1)\,\{4\,r\,(p\,r+q^2)\,(r^3+1) - 3\,r^4 q^2\}. \end{array}\right\} \quad (17)$$

Divide the first of these by the second, extract the square root, and divide by $p^3\,(r^3+1) - r^3\,(p^3+1)$. This gives

$$4\,(p^3+1)\,(r^3+1)\,(p\,r+q^2) = 3\,q^2\,\{p^3\,(r^3+1) + r^3\,(p^3+1)\}.$$

Substituting back in (16) and (17), we find

$$16\,(p^3+1)\,(r^3+1) = 9\,p^2 q^2 r^2 = \beta^2\gamma^2 = 4\,q^6.$$

Hence,

$$\frac{p^3}{p^3+1} + \frac{r^3}{r^3+1} = \frac{20}{9}, \qquad \frac{p^3\,r^3}{(p^3+1)\,(r^3+1)} = \frac{32}{27};$$

and therefore

$$\frac{p^3}{p^3+1} = \frac{4}{3}, \qquad \frac{r^3}{r^3+1} = \frac{8}{9},$$

so that

$$p = -\sqrt[3]{4}, \quad r = 2;$$

and therefore

$$q = \sqrt{-3}\,\sqrt[3]{2},$$

$$\alpha = \sqrt{-3}, \quad \beta = 3\,\sqrt[3]{4}, \quad \gamma = -2\,\sqrt{-3}\,\sqrt[3]{2}, \quad \delta = -3.$$

Hence,

$$p\,x^2 + q\,x\,y + r\,y^2 = -\sqrt[3]{4}\,x^2 + \sqrt{-3}\,\sqrt[3]{2}\,x\,y + 2\,y^2, \quad \text{(V)}$$

and 35 other solutions of the same type.

If we take the simplest solution

$$p\,x^2 + q\,x\,y + r\,y^2 \equiv -x^2,$$

then

$$F_2 \equiv \frac{-x^2}{\sqrt[3]{4}} \quad \text{and} \quad f_3 \equiv y^3.$$

Hence, from (12), $z = \frac{1}{2} f_1$ is bitangent to

$$\left(z^2 + \frac{3\,x^2}{\sqrt[3]{4}}\right)^2 - 4\,z\,y^3 - \frac{12\,x^4}{2\,\sqrt[3]{2}} = 0. \quad (18)$$

Writing x for $\dfrac{x}{\sqrt[3]{2}}$, this equation becomes

$$4z(z^3-y^3)-3(z^2-x^2)^2=0. \tag{19}$$

Hence, $z=0$, $z=y$, $\varepsilon^2 y$, $\varepsilon^4 y$ are bitangents.

To find the remaining 24 bitangents, consider the cubic surface

$$zu^2+\sqrt{3}\,(z^2-x^2)\,u+z^3-y^3=0. \tag{20}$$

The tangent cone vertex $V\,(0,0,0,1)$ meets $u=0$ in the quartic (19). The plane through V and any bitangent to (19) meets (20) in one of the 27 lines and a conic. Four planes of the pencil

$$u=k\,(z-y), \quad (k\neq 0), \tag{21}$$

through the line $u=0$, $z=y$ meet (20) in a triangle. Since

$$k^2(z-y)\,z+\sqrt{3}\,k\,(z^2-x^2)+z^2+yz+z^2=0 \tag{22}$$

passes through the intersection of (20) and (21), we can determine k by making

$$k^2(z-y)\,z+\sqrt{3}\,k\,z^2+z^2+yz+z^2=0$$

a perfect square. Hence,

$$(k^2-1)^2=4\,(k^2+\sqrt{3}\,k+1)\,;$$

that is,

$$(k+\sqrt{3}\,)\,(k^3-\sqrt{3}\,k^2-3\,k-\sqrt{3}\,)=0.$$

Hence,

$$k=-\sqrt{3}\,,\quad \frac{(1+\sqrt[3]{2}\,)^2}{\sqrt{3}},\quad \frac{(1+\varepsilon^2\sqrt[3]{2}\,)^2}{\sqrt{3}},\quad \frac{(1+\varepsilon^4\sqrt[3]{2}\,)^2}{\sqrt{3}}.$$

Substituting in (22), we find

$$y+\frac{1-k^2}{2}z=\sqrt{\sqrt{3}\,k}\cdot x\,.$$

Hence, the 28 bitangents of (18) are

$$z=0,\quad z=y,\quad z=y\pm\frac{\sqrt{-3}\,x}{\sqrt[3]{2}},$$

$$y-(1+\sqrt[3]{2}+\sqrt[3]{4}\,)\,z=\pm\frac{(1+\sqrt[3]{2}\,)\,x}{\sqrt[3]{2}},$$

$$y-(1+\varepsilon^2\sqrt[3]{2}+\varepsilon^4\sqrt[3]{4}\,)\,z=\pm\frac{(1+\varepsilon^2\sqrt[3]{2}\,)\,x}{\sqrt[3]{2}},$$

$$y-(1+\varepsilon^4\sqrt[3]{2}+\varepsilon^2\sqrt[3]{4}\,)\,z=\pm\frac{(1+\varepsilon^4\sqrt[3]{2}\,)\,x}{\sqrt[3]{2}},$$

and 18 others found by substituting $\varepsilon^2 y$ or $\varepsilon^4 y$ for y. Selecting $z=0$ as the simplest case, we have $f_1\equiv 0$. Hence, from (9) and (11),

$$f_2 + \phi_2 = \frac{3\,x^2}{\sqrt[3]{4}}, \qquad f_2 - \phi_2 = \frac{\sqrt{-3}\,x^2}{\sqrt[3]{4}},$$

$$f_2 = \frac{3 + \sqrt{-3}}{2\sqrt[3]{4}}\,x^2 = -\,\alpha\,\varepsilon^2\,x^2,$$

$$\phi_2 = \frac{3 - \sqrt{-3}}{2\sqrt[3]{4}}\,x^2 = \alpha\,\varepsilon^4\,x^2,$$

where $\alpha = \frac{\sqrt{-3}}{\sqrt[3]{4}}$. The cubic surface is therefore

$$z\,(z\,u - \alpha\,\varepsilon^2\,x^2) + u\,(z\,u + \alpha\,\varepsilon^4\,x^2) + y^3 = 0. \tag{23}$$

Theoretically, in order to determine the 27 lines on (23), we could find the intersection of (23) with the 120 quadrics derivable from the 120 conics previously found. Practically, it is easier to write (23) in the form

$$\alpha\,\varepsilon^2\,x^2\,(\varepsilon^2\,u - z) + y^3 + z^2\,u + z\,u^2 = 0. \tag{24}$$

The plane $z = \varepsilon^2\,u$ meets (24) in the 3 lines

$$z = \varepsilon^2\,u, \qquad y = u, \qquad \varepsilon^2\,u \text{ or } \varepsilon^4\,u.$$

Proceeding as in the determination of the bitangents to the quartic, we find that through the intersection of (24) and the plane

$$\varepsilon^2\,u - z = k\,(y - u) \tag{25}$$

passes the cone

$$\alpha\,\varepsilon^2\,k\,x^2 + y^2 + u\,y\,(1 + k^2) + (1 - k - 2\,k\,\varepsilon^2 - k^2)\,u^2 = 0. \tag{26}$$

This cone breaks up into 2 planes if

$$(1 + k^2)^2 = 4\,(1 - 2\,k\,\varepsilon^2 - k^2).$$

Hence,

$$k = -\sqrt{-3}, \quad \frac{(1 + \sqrt[3]{2}\,)^2}{-\sqrt{-3}}, \quad \frac{(1 + \varepsilon^2\,\sqrt[3]{2}\,)^2}{-\sqrt{-3}}, \quad \frac{(1 + \varepsilon^4\,\sqrt[3]{2}\,)^2}{-\sqrt{-3}}.$$

Substituting the values of k in (25) and (26), we find the 4 lines

$$\left\{\begin{aligned} y - u &= \pm\,\frac{\sqrt{-3}}{\sqrt[3]{2}}\,\varepsilon\,x, \\ y - \varepsilon^4\,z &= \pm\,\frac{\sqrt{-3}}{\sqrt[3]{2}}\,\varepsilon^3\,x, \end{aligned}\right.$$

$$\left\{\begin{aligned} y - (1 + \sqrt[3]{2} + \sqrt[3]{4}\,)\,u &= \pm\,\frac{(1 + \sqrt[3]{2}\,)\,\varepsilon\,x}{\sqrt[3]{2}}, \\ y - \varepsilon^4\,(1 + \varepsilon^2\,\sqrt[3]{2} + \varepsilon^4\,\sqrt[3]{4}\,)\,z &= \pm\,\frac{(1 + \varepsilon^2\,\sqrt[3]{2}\,)\,\varepsilon^3\,x}{\sqrt[3]{2}}, \end{aligned}\right.$$

and 4 similar lines found by changing $\sqrt[3]{2}$ into $\varepsilon^2\sqrt[3]{2}$ or $\varepsilon^4\sqrt[3]{2}$ except in the denominator of the right-hand members. By changing y into $\varepsilon^2 y$ or $\varepsilon^4 y$ we find the remaining 16 lines. For brevity, denote by $\sqrt{-3}$, $\sqrt[3]{2}$, $\varepsilon^2\sqrt[3]{2}$, $\varepsilon^4\sqrt[3]{2}$ the value of k used, by y, $\varepsilon^2 y$, $\varepsilon^4 y$ the value of y, and by $+x$, $-x$ the value of x. The 27 lines can now be chosen as follows:

$$\begin{array}{lll}
a_1,\ y,\ \sqrt{-3},\ x; & a_2,\ y,\ \sqrt[3]{2},\ x; & a_3,\ \varepsilon^2 y,\ \varepsilon^4\sqrt[3]{2},\ -x; \\
a_4,\ \varepsilon^2 y,\ \sqrt[3]{2},\ -x; & a_5,\ \varepsilon^4 y,\ \sqrt{-3},\ -x; & a_6,\ \varepsilon^4 y,\ \varepsilon^4\sqrt[3]{2},\ x; \\
b_1,\ y,\ \sqrt[3]{2},\ -x; & b_2,\ y,\ \sqrt{-3},\ -x; & b_3,\ \varepsilon^2 y,\ \sqrt[3]{2},\ x; \\
b_4,\ \varepsilon^2 y,\ \varepsilon^4\sqrt[3]{2},\ x; & b_5,\ \varepsilon^4 y,\ \varepsilon^4\sqrt[3]{2},\ -x; & b_6,\ \varepsilon^4 y,\ \sqrt{-3},\ x; \\
c_{13},\ \varepsilon^4 y,\ \varepsilon^2\sqrt[3]{2},\ -x; & c_{14},\ \varepsilon^4 y,\ \sqrt[3]{2},\ -x; & c_{15},\ \varepsilon^2 y,\ \varepsilon^2\sqrt[3]{2},\ x; \\
c_{24},\ \varepsilon^4 y,\ \varepsilon^2\sqrt[3]{2},\ x; & c_{23},\ \varepsilon^4 y,\ \sqrt[3]{2},\ x; & c_{26},\ \varepsilon^2 y,\ \varepsilon^2\sqrt[3]{2},\ -x; \\
c_{16},\ \varepsilon^2 y,\ \sqrt{-3},\ x; & c_{25},\ \varepsilon^2 y,\ \sqrt{-3},\ -x; & c_{35},\ y,\ \varepsilon^4\sqrt[3]{2},\ -x; \\
c_{36},\ y,\ \varepsilon^2\sqrt[3]{2},\ x; & c_{45},\ y,\ \varepsilon^2\sqrt[3]{2},\ -x; & c_{46},\ y,\ \varepsilon^4\sqrt[3]{2},\ x; \\
c_{12}\begin{cases}\varepsilon^2 z - u = 0 \\ y = u\end{cases}; & c_{34}\begin{cases}\varepsilon^2 u - z = 0 \\ y = \varepsilon^2 u\end{cases}; & c_{56}\begin{cases}\varepsilon^2 u - z = 0 \\ y = \varepsilon^4 u\end{cases}.
\end{array}$$

This choice was determined by first choosing c_{12}, c_{34}, c_{56}. Then a_1 was chosen, which fixes b_2; and a_2, which fixes b_1. The others were then found, first the a's, then the b's and c's. The transformation $y = \varepsilon y'$, $z = \varepsilon^5 z'$ in the double plane gives, from (4), $zu = z'u'$ on the cubic surface. Also, from (3),

$$z\,(zu + f_2) - u\,(zu + \phi_2) = \mp\,\{z'\,(z'u' + f_2') - u'\,(z'u' + \phi_2')\},$$

according as we do or do not interchange the sheets in the double plane.

From (1), since $\varepsilon^3 = -1$, we find

$$z\,(zu + f_2) + u\,(zu + \phi_2) = -\,\{z'\,(z'u' + f_2') + u'\,(z'u' + \phi_2')\}.$$

Hence, if we interchange the sheets in the double plane, the transformation is linear on the surface, and is

$$z = -z', \quad u = -u', \quad x = x', \quad y = \varepsilon y';$$

that is,

$$z = z', \quad u = u', \quad x = -x', \quad y = \varepsilon^4 y'.$$

This transformation sends a_1 into a_5, a_2 into c_{14}, a_3 into c_{46}, a_4 into a_2, a_5 into c_{16}, a_6 into a_3. It leaves O and O' invariant. It is therefore a quadratic transformation of Kantor's B_6 type "b' en c, c' en a, a' en b," a, b, c, a', b', c' being a_2, a_3, a_5, a_6, a_1, a_4, respectively. If we do not interchange the sheets,

the transformation is of degree 16 having a_1, a_4, a_6, O, O' for 6-fold points, and a_2, a_3, a_5 for 5-fold points.

The transformation $x = \varepsilon x'$ in the double plane gives

$$z u + \frac{x^2}{\sqrt[3]{4}} = z' u' + \frac{x'^2}{\sqrt[3]{4}}$$

on the cubic surface. Hence,

$$z u = z' u' + \phi_2' .$$

Also

$$z (z u + f_2) + u (z u + \phi_2) = - f_3 = z' (z' u' + f_2') + u' (z' u' + \phi_2'),$$

and

$$z (z u + f_2) - u (z u + \phi_2) = \mp \{ z' (z' u' + f_2') - u' (z' u' + \phi_2') \},$$

according as we do or do not interchange the sheets on the double plane. If we do interchange,

$$z = \frac{u' (z' u' + \phi_2')}{z u + f_2} = \frac{u' (z' u' + \phi_2')}{z' u'},$$

$$u = \frac{z' (z' u' + f_2')}{z u + \phi_2} = \frac{z' (z' u' + f_2')}{z' u' + f_2'} .$$

Hence,

$$x = \varepsilon x' \cdot z', \qquad y = y' \cdot z', \qquad z = u' (z' u' + \phi_2'), \qquad u = z'^2.$$

This is a quadric transformation, the image of $O (0, 0, 1, 0)$ being the plane $z' = 0$. Any plane through O is transformed into a plane through O'. Hence, the image of any line on the cubic surface is found by first transforming each line by changing the first plane on which it lies by $u = z'$, $x = \varepsilon x'$ into the second plane on which another line lies. This is equivalent to changing y into $\varepsilon^4 y'$, x into $-x'$. The image of the first line is then the residual conic found by passing a plane through the second line and O'. Hence,

a_1 is transformed into a conic residual to a_5,
a_2 " " " " " " " c_{14},
a_3 " " " " " " " c_{46},
a_4 " " " " " " " a_2,
a_5 " " " " " " " c_{16},
a_6 " " " " " " " a_3.

O is transformed into $z' = 0$, the section of the cubic surface by the tangent plane at O'. O' is transformed into O. Hence, in the plane,

a_1	is transformed	into	the	cubic	$\begin{pmatrix} a_1, & a_2, & a_3, & a_4, & a_5, & a_6, & O, & O' \\ 1, & 1, & 1, & 1, & 2, & 1, & 0, & 1 \end{pmatrix}$,
a_2	"	"	"	conic	$(0, 1, 1, 0, 1, 1, 0, 1)$,
a_3	"	"	"	conic	$(1, 1, 1, 0, 1, 0, 0, 1)$,
a_4	"	"	"	cubic	$(1, 2, 1, 1, 1, 1, 0, 1)$,
a_5	"	"	"	conic	$(0, 1, 1, 1, 1, 0, 0, 1)$,
a_6	"	"	"	cubic	$(1, 1, 2, 1, 1, 1, 0, 1)$,
O	"	"	"	cubic	$(1, 1, 1, 1, 1, 1, 0_2\ 2)$,
O'	"	"	"	the point O.	

This is a transformation of degree 7. If we do not interchange the sheets, we find a transformation of degree 11 having 4 3-fold, 3 4-fold and 1 6-fold point.

CORNELL UNIVERSITY, *November*, 1913.